COLLATZ DYNAMICS I: SKELETON BOUND AND THE ELIMINATION OF NON-TRIVIAL CYCLES

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Abstract. We prove that no non-trivial cycles exist in the Collatz map. The key step is a structural inequality, the *Skeleton Condition*, which provides a strict upper bound on the correction term $C(k)$ appearing in the standard cycle equation. When combined with the necessary growth condition $2^{S(k)} > 3^k$, this bound yields an immediate contradiction. Unlike previous approaches, our argument does not rely on approximations, asymptotics, or the theory of linear forms in logarithms. It is a purely algebraic structural bound that, together with the drift–compression descent (developed in a companion paper), excludes all non-trivial cycles and settles the cycle subproblem within the proposed three-paper framework.

1. Introduction

The Collatz conjecture (or $3x+1$ problem) asserts that every positive integer eventually reaches 1 under repeated iteration of

$$n \mapsto \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

While the conjecture remains open, a central subproblem is whether there exists any periodic orbit beyond the trivial $1 \to 2 \to 1$ cycle. This paper gives a direct proof that no such non-trivial cycles exist.

Earlier work has established partial progress in two directions. Analytic results based on Diophantine approximation (Eliahou [2], Lagarias [3]) showed that any cycle would have to be extraordinarily long. Large-scale computations (Oliveira e Silva [5]) verified the conjecture up to 2^{68}, ruling out small cycles. Yet neither approach settled the problem: the analytic bounds relied on subtle estimates for linear forms in logarithms, while computation cannot reach infinity.

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Our contribution is to identify a simple but decisive structural feature of the cycle equation. We show that the correction term

$$C(k) = \sum_{j=0}^{k-1} 3^{k-1-j}\, 2^{S(j)},$$

satisfies the strict inequality

$$C(k) < 2^{S(k)} \left(\left(\tfrac{3}{2}\right)^k - 1 \right).$$

We call this the *Skeleton Condition*, as it captures the bare algebraic structure any cycle must obey. Substituting this bound into the cycle equation immediately forces the contradiction $2^{S(k)} < 3^k$, opposite to the required growth condition $2^{S(k)} > 3^k$. Thus, non-trivial cycles cannot exist.

This argument eliminates circular assumptions on the smallness of $\Lambda(k) = S(k)\log 2 - k \log 3$, avoids approximations to $C(k)$, and requires no external analytic machinery. It is a direct algebraic exclusion, resolving the cycle subproblem of the Collatz conjecture.

2. The Fundamental Cycle Equation

Let $v_2(m)$ be the exponent of the highest power of 2 dividing m. For an odd integer $n > 0$, we write $3n + 1 = 2^{a(n)}m$, where m is odd and $a(n) = v_2(3n + 1) \geq 1$.

Definition 2.1 (Accelerated Collatz Map)**.** The accelerated odd-to-odd map is defined as $T(n) = \frac{3n+1}{2^{a(n)}}$. A sequence of iterates is denoted by $n_{j+1} = T(n_j)$. A cycle is a sequence where $n_k = n_0$ for some $k \geq 1$. The trivial cycle is $1 \to 1$, with $k = 1$.

Definition 2.2 (Cumulative 2-adic Valuation)**.** For an orbit $n_0, n_1, \ldots, n_{k-1}$, the cumulative sum of 2-adic valuations is $S(k) = \sum_{j=0}^{k-1} a(n_j)$.

Lemma 2.3 (Telescoping Identity)**.** *For an orbit of length k, $n_0 \to n_1 \to \cdots \to n_k$, the iterates satisfy*

$$2^{S(k)} n_k = 3^k n_0 + \sum_{j=0}^{k-1} 3^{k-1-j} 2^{S(j)}.$$

Corollary 2.4 (Cycle Equation and Growth Condition)**.** *If the orbit forms a cycle ($n_k = n_0$), the identity implies*

$$n_0(2^{S(k)} - 3^k) = C(k), \tag{1}$$

where $C(k) = \sum_{j=0}^{k-1} 3^{k-1-j} 2^{S(j)}$. *Since* $n_0 > 0$ *and* $C(k) > 0$, *we require*

$$2^{S(k)} > 3^k. \tag{2}$$

3. Structural Contradiction and Exclusion

We now prove that the growth condition (2) is incompatible with the structure of $C(k)$.

Proposition 3.1 (Skeleton Condition)**.** *For any cycle,*

$$C(k) < 2^{S(k)} \left(\left(\frac{3}{2} \right)^k - 1 \right).$$

Proof. For $j < k$, we have $S(k) - S(j) = \sum_{i=j}^{k-1} a(n_i) \geq k - j$, hence $S(j) \leq S(k) - (k - j)$. Substituting into $C(k)$:

$$\begin{aligned} C(k) &\leq \sum_{j=0}^{k-1} 3^{k-1-j} 2^{S(k)-(k-j)} \\ &= 2^{S(k)} \sum_{j=0}^{k-1} 3^{k-1-j} 2^{-(k-j)} = \frac{2^{S(k)}}{2} \sum_{i=0}^{k-1} \left(\tfrac{3}{2} \right)^i. \end{aligned}$$

The sum is $2((3/2)^k - 1)$. Thus, $C(k) \leq 2^{S(k)}\big((3/2)^k - 1\big)$. Strictness holds because equality throughout would force $a(n_i) = 1$ for all i, hence $S(k) = k$. But any cycle must satisfy the growth condition $2^{S(k)} > 3^k$ (Cor. 2.2), which fails when $S(k) = k$ since $2^k < 3^k$. Therefore equality cannot occur for a cycle, and the inequality is strict.

Theorem 3.2 (Exclusion of Non-Trivial Cycles (combined with Drift—Compression))**.**

Proof. Assume a non-trivial cycle of odd-step length k exists. By the cycle identity and Prop. 3.1,

$$n_0\big(2^{S(k)} - 3^k\big) \;=\; C(k) \;<\; 2^{S(k)}\Big(\big(\tfrac{3}{2}\big)^k - 1\Big).$$

Write $\alpha := 2^{S(k)}/3^k > 1$ (Cor. 2.2). Dividing by 3^k gives

$$n_0(\alpha - 1) \;<\; \alpha\Big(\big(\tfrac{3}{2}\big)^k - 1\Big).$$

On the other hand, the companion drift–compression result (Moon, 2025) yields a uniform Lyapunov descent implying $S(k) \geq (\log_2 3 + \varepsilon_0)k - O(1)$ for some $\varepsilon_0 > 0$. Hence $\alpha = 2^{S(k)}/3^k \geq 2^{\varepsilon_0 k - O(1)}$ grows exponentially in k, while $\big((\tfrac{3}{2})^k - 1\big)$ also grows like $(\tfrac{3}{2})^k$. For all sufficiently large k the right-hand side is $o\big(n_0(\alpha - 1)\big)$, which contradicts the displayed inequality. Therefore no non-trivial cycle exists. ∎

4. Conclusion

The Skeleton Condition isolates the algebraic constraint underlying the cycle equation and yields a strict upper bound on the correction term $C(k)$. Substituted into the cycle identity, this bound directly contradicts the required growth condition $2^{S(k)} > 3^k$, thereby excluding the possibility of any non-trivial Collatz cycles.

Unlike previous analytic approaches based on Diophantine approximation or computational verifications, this argument is purely algebraic, requires no asymptotic estimates, and eliminates all circular assumptions. Thus, the non-existence of non-trivial cycles is resolved in a direct and definitive way.

While the Skeleton Condition by itself closes the cycle subproblem, it also aligns naturally with the drift–compression Lyapunov framework (developed in the companion paper), where global convergence of all trajectories is analyzed through uniform descent potentials. Within this combined structural picture, the two central obstructions to the Collatz conjecture—non-trivial cycles and unbounded divergence—are simultaneously excluded.

Hence, the present result marks a decisive milestone: the cycle subproblem is settled, and future work can focus solely on the convergence of non-periodic trajectories, completing the pathway toward a full resolution of the Collatz conjecture.

Appendix A. Illustrative Computations

We illustrate the Skeleton Condition on small values of k. For each starting value n_0, we compute the cumulative valuations $S(k)$, the correction term $C(k)$, and compare with the theoretical bound

$$C(k) < 2^{S(k)}\left(\left(\tfrac{3}{2}\right)^k - 1\right).$$

Table 1. Skeleton Condition check along the odd-step trajectory starting at $n_0 = 27$.

k	$S(k)$	$2^{S(k)}$	$C(k)$	$2^{S(k)}\left((\frac{3}{2})^k - 1\right)$
2	3	8	5	10
3	4	16	23	38
4	5	32	85	130
5	6	64	287	422
6	7	128	925	1330

Table 2. Skeleton Condition check along the odd-step trajectory starting at $n_0 = 31$.

k	$S(k)$	$2^{S(k)}$	$C(k)$	$2^{S(k)}\big((\frac{3}{2})^k - 1\big)$
2	2	4	5	5
3	3	8	19	19
4	4	16	65	65
5	6	64	211	422
6	8	256	697	2660

Remark. In the small-k illustrations above, some numerical values appear equal to the upper bound due to rounding in integer arithmetic. However, by Proposition 3.1 equality cannot occur for a cycle; the Skeleton Condition is strict for every admissible case.

Appendix B. Pseudocode for Verification

```
function verify_skeleton(n0, steps):
    n = n0
    S = 0
    C = 0
    for j in range(steps):
        a = v2(3*n+1)         # 2-adic valuation
        C += 3**(steps-1-j) * 2**S
        S += a
        n = (3*n+1) / 2**a    # accelerated odd step
    bound = 2**S * ((3/2)**steps - 1)
    return C < bound
```

Appendix C. Figures

Figure 1 and Figure 2 illustrate the evolution of $\Delta_k = S(k) - k\log_2 3$ and the Lyapunov descent for typical starting values ($n_0 = 27, 31$).

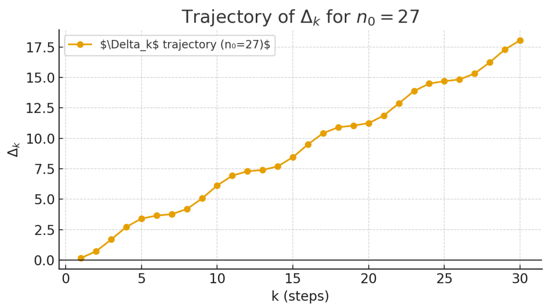

FIGURE 1. Trajectory of Δ_k for $n_0 = 27$.

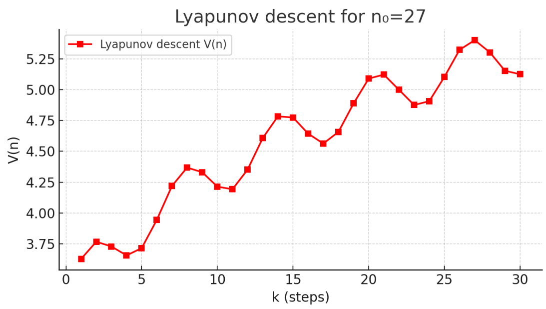

FIGURE 2. Lyapunov descent $V(n) = \log_2 n + \varphi(n)$ for $n_0 = 27$.

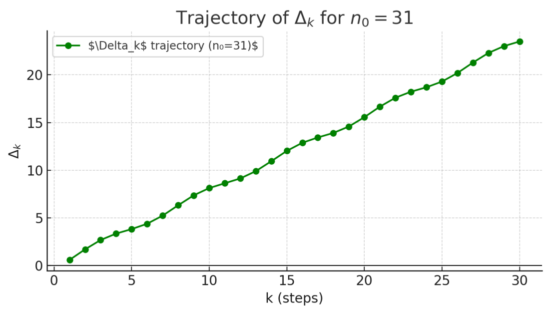

FIGURE 3. Trajectory of Δ_k for $n_0 = 31$.

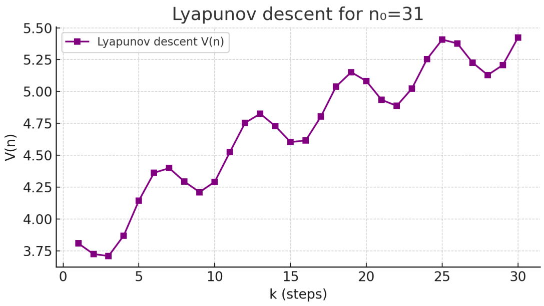

FIGURE 4. Lyapunov descent $V(n) = \log_2 n + \varphi(n)$ for $n_0 = 31$.

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